

THE COHERENCE THEOREM FOR ANN-CATEGORIES

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February 1, 2008

abstract

This paper¹ presents the proof of the coherence theorem for Ann-categories whose set of axioms and original basic properties were given in [9]. Let

$$\mathcal{A} = (\mathcal{A}, \mathfrak{A}, c, (0, g, d), a, (1, l, r), \mathfrak{L}, \mathfrak{R})$$

be an Ann-category. The coherence theorem states that in the category \mathcal{A} , any morphism built from the above isomorphisms and the identification by composition and the two operations \otimes , \oplus only depends on its source and its target.

The first coherence theorems were built for monoidal and symmetric monoidal categories by Mac Lane [7]. After that, as shown in the References, there are many results relating to the coherence problem for certain classes of categories.

For Ann-categories, applying Hoang Xuan Sinh's ideas used for Gr-categories in [2], the proof of the coherence theorem is constructed by faithfully "embedding" each arbitrary Ann-category into a quite strict Ann-category. Here, a *quite strict* Ann-category is an Ann-category whose all constraints are strict, except for the commutativity and left distributivity ones.

This paper is the work continuing from [9]. If there is no explanation, the terminologies and notations in this paper mean as in [9].

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1 Canonical isomorphisms

In this section, we define some canonical isomorphisms induced by isomorphisms $c, \mathfrak{L}, \widehat{L}^A$ and the identification, laws \otimes, \oplus on the quite strict Ann-category \mathcal{A} .

Let I be a fully ordered limited set. If $I \neq \emptyset$ and α is the maximal of I , we will denote $I' = I \setminus \{\alpha\}$; and the notation $|I|$ refers to the cardinal of I .

Definition 1.1. [1] *The canonical sum $\sum_I A_i$ where $A_i \in \text{Ob}\mathcal{A}, i \in I$ is defined inductively as follows*

1. $\sum_I A_i = 0$ if $I = \emptyset$ and $\sum_I A_i = A_\alpha$ if $I = \{\alpha\}$.
2. $\sum_I A_i = (\sum_{I'} A_i \oplus A_\alpha)$ if $|I| > 1$.

Definition 1.2. *We define the isomorphism*

$$\nu_{\sum A_i, \sum B_i} : (\sum_I A_i) \oplus (\sum_I B_i) \longrightarrow \sum_I (A_i \oplus B_i),$$

(which is abbreviated by ν_I) by induction on $|I|$ as follows

1. $\nu_I = \text{id}$ if $|I| \leq 1$.
2. if $|I| > 2$, ν_I is defined by the following commutative diagram

¹This paper has been published (in Vietnamese) in Vietnam Journal of Mathematics Vol. XVI, No 1, 1988.

$$\begin{array}{ccc}
(\sum_I A_i) \oplus (\sum_I B_i) & \xrightarrow{id} & (\sum_{I'} A_i) \oplus A_\alpha \oplus (\sum_{I'} B_i) \oplus B_\alpha \\
\nu_I \downarrow & & \downarrow \nu \\
\sum_I (A_i \oplus B_i) & \xrightarrow{\nu_{I'} \oplus id_{A_\alpha \oplus B_\alpha}} & (\sum_{I'} A_i) \oplus (\sum_{I'} B_i) \oplus A_\alpha \oplus B_\alpha
\end{array}$$

where $\nu = id \oplus c \oplus id$. We can see that the isomorphism ν_I is built only from isomorphisms c, id by law \oplus . Moreover, the isomorphism ν_I is natural.

Definition 1.3. We define following isomorphisms

$$u_{I,J} : \sum_I \sum_J (A_i \otimes B_j) \longrightarrow \sum_J \sum_I (A_i \otimes B_j)$$

by induction on $|I|$ as follows

1. $u_{I,J} = id$ if $|I| \leq 1$ or $J = \emptyset$.

2. $u_{I,J} = \nu_J(u_{I',J} \oplus id)$ if $|I| > 1$.

So, isomorphisms $u_{I,J}$ are also built from the isomorphisms c, id by law \oplus and these morphisms are functorial.

Definition 1.4. We define following isomorphisms

$$F_{I,J} : (\sum_I A_i) \otimes (\sum_J B_j) \longrightarrow \sum_I \sum_J (A_i \otimes B_j) = \sum_{I \times J} (A_i \otimes B_j)$$

where $I \times J$ is odered alphabetically as follows

1. $F_{I,J} = id : 0 \otimes (\sum_J B_j) \rightarrow 0$ if $I = \emptyset$.

(since $\mathfrak{R} = id$ we have $\widehat{R}^A = id$ for all $A \in Ob\mathcal{A}$)

2. $F_{I,J} = \widehat{L}^X : X \otimes 0 \rightarrow 0$ where $X = \sum_I A_i$ if $J = \emptyset$.

3. $F_{I,J} = \sum_I f_{A_i}$ if $I \neq \emptyset$ and $J \neq \emptyset$, where

$$f_A : A \otimes (\sum_J B_j) \rightarrow \sum_J A \otimes B_j$$

is defined as follows: If $|J| = 1$, $f_A = id$; whereas f_A is defined by induction on $|J|$ by the following commutative diagram

$$\begin{array}{ccc}
A \otimes (\sum_J B_j) & \xrightarrow{f_A} & (\sum_J (A \otimes B_j)) \\
\downarrow \widehat{L}^A & & \parallel \\
A \otimes (\sum_J B_j) \oplus (A \otimes B_\beta) & \xrightarrow{f'_A \otimes id} & (\sum_{J'} A \otimes B_j) \oplus (A \otimes B_\beta)
\end{array}$$

where β is the maximal element of J and $J' = J \setminus \{\beta\}$.

Definition 1.5. We define following isomorphisms

$$K_{I,J} : (\sum_I A_i) \otimes (\sum_J B_j) \longrightarrow \sum_J \sum_I (A_i \otimes B_j)$$

as follows

1. $K_{I,J} = id$ if $I = \emptyset$.

2. $K_{I,J} = \widehat{L}^X$, where $X = \sum_I A_i$ if $J = \emptyset$.

3. $K_{I,J} = f_X$, where $X = \sum_I A_i$ in other cases.

Then, we have the following proposition immediately

Proposition 1.6. *With canonical sums $\sum_I A_i, \sum_J A_j$, we have the relation*

$$K_{I,J} = u_{I,J} \cdot F_{I,J}.$$

Applying this proposition we can prove

Proposition 1.7. *Assume J_1, J_2 be non-empty subsets of J such that $J = J_1 \coprod J_2$ and $j_1 < j_2$ if $j_1 \in J_1, j_2 \in J_2$. Then for sums $A = \sum_I A_i, \sum_J B_j, \sum_{J_1} B_j, \sum_{J_2} B_j$ we have following relations*

$$\begin{aligned} F_{I,J} &= \nu_I \cdot (F_{I,J_1} \oplus F_{I,J_2}) \cdot \check{L}^A \\ F_{J,I} &= F_{J_1,I} \oplus F_{J_2,I}. \end{aligned}$$

We will give the proof of this proposition in detail to illustrate the proof using commutative diagrams. Hereafter, for convenience, we write AB instead of $A \otimes B$ for all $A, B \in \text{Obs}(\mathcal{A})$.

Proposition 1.8. *In the Ann-category \mathcal{A} , the following diagrams*

$$\begin{array}{ccc} (\sum_I A_i) \otimes (\sum_J B_j) \otimes (\sum_T C_t) & \xrightarrow{F_{I,J} \otimes id} & (\sum_{I \times J} A_i \otimes B_j) \otimes (\sum_T C_t) \\ \downarrow id \otimes F_{J,T} & & \downarrow F_{I \times J, T} \\ (\sum_I A_i) \otimes (\sum_{J \times T} B_j \otimes C_t) & \xrightarrow{F_{I, J \times T}} & \sum_{I \times J \times T} A_i \otimes B_j \otimes C_t \end{array}$$

commute.

Proof. 1. In case $I = \emptyset$, we have the proposition proved since the diagram (1.1) becomes the following one

$$\begin{array}{ccc} 0(\sum_J B_j)(\sum_T C_t) & \xrightarrow{id \otimes id} & 0(\sum_T C_t) \\ id \otimes F_{J,T} \downarrow & & \downarrow id \\ 0(\sum_{J \times T} B_j \otimes C_t) & \xrightarrow{id} & 0 \end{array}$$

whose commutativity follows from $\hat{R}^X = id$ and the property of the zero object (see Prop.3.2 [9]). In case $J = \emptyset$ or $T = \emptyset$ the proposition is proved similarly. Hence, we now can suppose that I, J, T are all not empty.

2. In case $|I| = 1$. Firstly, consider the case in which $|J| = 1$. We prove the proposition by induction on $|T|$. Then

If $|T| = 1$, the proof is obvious.

If $|T| = 2$, the diagram commutes thanks to the axiom (1.1) of Ann-categories (see [9]).

If $|T| > 2$, consider the diagram (1.2).

$$\begin{array}{ccccc}
A(B \sum_{T'} C_t \oplus BC_\gamma) & \xrightarrow{\check{L}^A} & AB(\sum_T C_t) \oplus ABC_\gamma & & \\
\downarrow id \otimes \check{L}^B & & \downarrow id & & \\
AB(\sum_T C_t) & \xrightarrow{\check{L}^{AB}} & AB(\sum_{T'} C_t) \oplus ABC_\gamma & & \\
\downarrow id & & \downarrow f_{AB} \oplus id & & \\
AB(\sum_T C_t) & \xrightarrow{f_{AB}} & \sum_{T'} ABC_t \oplus ABC_\gamma & & \\
\downarrow id \otimes (f'_B \oplus id) & & \downarrow id & & \\
A(\sum_{T'} BC_t \oplus BC_\gamma) & \xrightarrow{f_A} & \sum_{T'} ABC_t \oplus ABC_\gamma & & \\
\uparrow id & & \uparrow f'_A \oplus id & & \\
A(\sum_{T'} BC_t \oplus BC_\gamma) & \xrightarrow{\check{L}^A} & A(\sum_{T'} BC_t) \oplus ABC_\gamma & &
\end{array}
\quad (2)$$

In this diagram, the region (I) commutes thanks to the axiom (1.1) in [9]; regions (II), (IV), (V) commute thanks to definitions of isomorphisms f_{AB} , f_A , f_B ; (VI) commute thanks to the inductive supposition; the parameter commutes since \check{L}^A is a functorial isomorphism. Therefore, the region (III) commutes. This completes the proof.

After that, still with the condition $|I| = 1$, we can prove the proposition with $|J| > 1$ by induction on $|J|$.

3. Now if $|I| > 1$, consider the diagram (1.3). In this diagram, the region (I) commute since $\mathfrak{R} = id$ is a functorial isomorphism; the region (II) commutes thanks to the inductive supposition for the first component of sums, for the second component since the case $|I| = 1$ has just been aproved above; the region (III) commutes thanks to the property of the isomorphism $F_{I,J}$ (see the Prop.1.7); regions (IV), (V) commute thanks to definitions of $F_{I,J}$ and $F_{I,J \times T}$. So the parameter commutes. This completes the proof. \square

$$\begin{array}{ccc}
 (\sum_I A_i)(\sum_J B_j)(\sum_T C_t) & \xrightarrow{id \otimes F_{J,T}} & (\sum_I A_i)(\sum_{J \times T} B_j C_t) \\
 \downarrow id & & \downarrow id \\
 (\sum_I A_i)(\sum_J B_j)(\sum_T C_t) \oplus A_\alpha(\sum_J B_j)(\sum_T C_t) & \xrightarrow{(id \otimes F_{J,T}) \oplus (id \otimes F_{J,T})} & (\sum_{I'} A_i)(\sum_{J \times T} B_j C_t) \oplus A_\alpha(\sum_{J \times T} B_j C_t) \\
 \downarrow (F_{I',J} \otimes id) \oplus (F_{A_\alpha} \otimes id) & & \downarrow F_{I',J \times T} \oplus f_{A_\alpha} \\
 (\sum_{I' \times J} A_i B_j)(\sum_T C_t) \oplus (\sum_J A_\alpha B_j)(\sum_T C_t) & \xrightarrow{F_{I' \times J,T} \oplus F_{J,T}} & \sum_{I' \times J \times T} A_i B_j C_t \oplus \sum_{J \times T} A_\alpha B_j C_t \\
 \downarrow id & & \downarrow id \\
 (\sum_{I' \times J} A_i B_j \oplus \sum_J A_\alpha B_j)(\sum_T C_t) & \xrightarrow{F_{I \times J,T}} & \sum_{I \times J \times T} A_i B_j C_t
 \end{array}$$

$F_{I,J} \otimes id$ $F_{I',J \times T} \oplus f_{A_\alpha}$ $F_{I,J \times T}$

2 The coherence theorem for Ann-categories

Let \mathcal{A} be a quite strict Ann-category. Assume $X_s, s \in \Omega$ be a non-empty, limited family of objects of \mathcal{A} and Y be an expression of the family X_s with operations \otimes and \oplus . With distributivity constraints $\mathfrak{L}, \mathfrak{R} = id$, induced isomorphisms \widehat{L}^A and isomorphisms $\mathfrak{A} = id, g, d = id, a = id, l, r = id$, we can write Y as a sum of *monomials* of objects of X_s by using the following isomorphism

$$h : Y \rightarrow \sum_I A_i$$

where $A_i \neq 0$ for all i if $I \neq \emptyset$ and h is built from the identification, and isomorphisms $\mathfrak{L}, \widehat{L}^A$. A such a pair $(h, \sum_I A_i)$ is called an *expansion form* of Y . We now define a *canonical expansion form* of Y by induction on its length, where the length of expansion form Y is the total number of times of appearances of objects A_i in Y . It is easily to see that any Y whose length is more than 1 can be written in the form of $U \otimes V$ or $U \oplus V$. That implies

Definition 2.1. *The canonical expansion form*

$$h : Y \rightarrow \sum_I A_i$$

of Y is defined as follows

1. If $Y = \sum_I X_i$, $h = id : \sum_I X_i \rightarrow \sum_I X_i$, where I is a subset of Ω ; whereas I' is the set of indexes i such that $X_i \neq 0$.
2. If $Y = U \otimes V$, the isomorphism h is the composition

$$Y = U \otimes V \xrightarrow{u \otimes v} (\sum_J B_j) \otimes (\sum_T C_t) \xrightarrow{F_{J,T}} \sum_{J \times T} B_j C_t$$

where $(u, \sum_J B_j), (v, \sum_T C_t)$ are, respectively, canonical expansion forms of U, V and defined by the induction supposition.

3. If $Y = U_1 \oplus U_2$, h is the composition

$$Y = U_1 \oplus U_2 \xrightarrow{u_1 \oplus u_2} (\sum_{I_1} B_i) \otimes (\sum_{I_2} B_i) \xrightarrow{id} \sum_{I'} B_i$$

where u_1, u_2 are defined by the inductive supposition; $I = I_1 \amalg I_2$; $i_1 < i_2$ if $i_1 \in I_1, i_2 \in I_2$ and I' is the set of indexes $i \in I$ such that $B_i \neq 0$.

Proposition 2.2. *If Y_1, Y_2 are expressions of objects $X_s, s \in \Omega$ and $\varphi : Y_1 \rightarrow Y_2$ is the morphism built from morphisms $c, \mathfrak{L}, \widehat{L}^A$, the identification and laws \otimes, \oplus together with the composition, then they can be embedded into the following commutative diagram*

$$\begin{array}{ccc} Y_1 & \xrightarrow{h_1} & \sum_I A_i \\ \varphi \downarrow & & \downarrow u \\ Y_2 & \xrightarrow{h_2} & \sum_I A_{\sigma(i)} \end{array}$$

where $(h_1, \sum_I A_i), (h_2, \sum_I A_{\sigma(i)})$ are, respectively, canonical expansion forms of Y_1, Y_2 ; σ is a permutation of the set I and u is an isomorphism built from the morphism c , the identification, the law \oplus together with the composition.

Proof. We can prove the proposition in case φ is one of isomorphisms $c, \mathfrak{L}, \widehat{L}^A, \mathfrak{A} = id, g = d = id, a = id, l = r = id$. Next, we can prove it easily in case φ is the sum \otimes or the product \oplus of two morphisms of above-mentioned ones. \square

We now state the coherence theorem.

Theorem 2.3. *Let Y_1, Y_2, \dots, Y_n be expressions of the family $X_s, s \in \Omega$ of objects in the quite strict Ann-category \mathcal{A} . Let $\varphi_{i,i+1} : Y_i \rightarrow Y_{i+1}$ ($i=1, 2, \dots, n$), $\varphi_{n,1} : Y_n \rightarrow Y_1$ be isomorphisms built from morphisms $c, \mathfrak{L}, \widehat{L}^A$, identification by laws \otimes, \oplus and the composition. Then, the following diagram*

$$\begin{array}{c} Y_1 \xrightarrow{\varphi_{1,2}} Y_2 \xrightarrow{\varphi_{2,3}} Y_3 \longrightarrow \dots \longrightarrow Y_n \\ \boxed{\hspace{10cm}} \\ \varphi_{n,1} \end{array} \quad (3)$$

commutes.

Proof. Let $(h_i, \sum_j A_{\sigma_i(j)})$ denote the canonical expansion form of Y_i . Consider the following diagram

$$\begin{array}{ccccccc} & & \xrightarrow{\hspace{10cm}} \varphi_{n,1} \xrightarrow{\hspace{10cm}} & & & & \\ & \downarrow & & & \downarrow & & \\ Y_1 & \xrightarrow{\varphi_{1,2}} & Y_2 & \longrightarrow & \dots & \longrightarrow & Y_{n-1} \xrightarrow{\varphi_{n-1,n}} Y_n \\ & \downarrow h_1 & \downarrow h_2 & & & & \downarrow h_{n-1} & \downarrow h_n \\ \sum_j A_{\sigma_1(j)} & \xrightarrow{u_{1,2}} & \sum_j A_{\sigma_2(j)} & \longrightarrow & \dots & \longrightarrow & \sum_j A_{\sigma_{n-1}(j)} & \xrightarrow{u_{n-1,n}} & \sum_j A_{\sigma_n(j)} \\ & & & & & & \xrightarrow{\hspace{10cm}} u_{n,1} \xrightarrow{\hspace{10cm}} & & \end{array} \quad (4)$$

Then we have the diagram (2.2), where morphisms $u_{i,i+1}$ ($i = 1, 2, \dots, n-1$), $u_{n,1}$ make regions (1), ..., (n) and the parameter commute according to Prop.2.2. They are built from e, id and the laws \otimes , so according to the coherence theorem for a symmetric monoidal category, the region (b) commutes. Therefore, the region (a) commutes. This completes the proof. \square

3 The general case

In this last section, we assume that $\mathcal{A}, \mathcal{A}'$ are Ann-categories with, respectively, constraints

$$\begin{aligned} & (\mathfrak{A}, c, (0, g, d), a, (1, l, r), \mathfrak{L}, \mathfrak{R}) \\ & (\mathfrak{A}', c', (0', g', d'), a', (1', l', r'), \mathfrak{L}', \mathfrak{R}') \end{aligned}$$

and $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$ is a faithful Ann-functor such that the pair (F, \tilde{F}) is compatible with the unitivity constraints $(1, l, r), (1', l', r')$. In addition, let $\check{F} : F1 \rightarrow 1'$ denote the isomorphism induced by the above compatibility.

Let $(X_i), i \in I$ be a non-empty, limited family of objects of \mathcal{A} , and $Y = \mathcal{H}(X_i)$ be a certain expression of the family $(X_i), i \in I$. Then, the expression $Y' = \mathcal{H}(X'_i)$ is called the *canonical*

image of $Y = \mathcal{H}(X_i)$ under F if

$$\begin{aligned} X'_i &= 0' && \text{when } X_i = 0 \\ X'_i &= 1' && \text{when } X_i = 1 \\ X'_i &= FX_i && \text{otherwise} \end{aligned}$$

From this notion we give the following definition

Definition 3.1. We define a canonical isomorphism

$$f : FY = F(\mathcal{H}(X_i)) \rightarrow F(\mathcal{H}(X'_i))$$

by induction on Y 's length as follows

1. If Y 's length is equal to 1, $Y = X_\alpha$ then

$$\begin{aligned} f &= \hat{F} : F0 \rightarrow 0' && \text{in case } X_\alpha = 0 \\ f &= \hat{F}' : F1 \rightarrow 1' && \text{in case } X_\alpha = 1 \\ f &= id : FX_\alpha \rightarrow FX_\alpha && \text{in other cases} \end{aligned}$$

2. If Y 's length is more than 1, $Y = U_1 \otimes U_2$ or $Y = U_1 \oplus U_2$. Then, the isomorphism f is, respectively, the following compositions

$$FY = F(U_1 \otimes U_2) \xrightarrow{\tilde{F}} FU_1 \otimes FU_2 \xrightarrow{f_1 \otimes f_2} \mathcal{H}_1(X'_1) \otimes \mathcal{H}_2(X'_2)$$

$$FY = F(U_1 \oplus U_2) \xrightarrow{\check{F}} FU_1 \oplus FU_2 \xrightarrow{f_1 \oplus f_2} \mathcal{H}_1(X'_1) \oplus \mathcal{H}_2(X'_2)$$

where f_1, f_2 are canonical isomorphisms determined by inductive supposition.

Proposition 3.2. Suppose that $\varphi : Y_1 \rightarrow Y_2$ is a morphism built from isomorphisms $\mathfrak{A}, c, g, d, a, l, r, \mathfrak{L}, \mathfrak{R}, \hat{L}^A, \hat{R}^A$ in the Ann-category \mathcal{A} . Then, φ can be embedded into the following commutative diagram

$$\begin{array}{ccc} FY_1 & \xrightarrow{f_1} & Y'_1 \\ F(\varphi) \downarrow & & \downarrow \varphi' \\ FY_2 & \xrightarrow{f_2} & Y'_2 \end{array}$$

where f_i are canonical isomorphisms corresponding to canonical images Y'_i of Y_i ($i=1, 2$), whereas φ' is a morphism built from isomorphisms $\mathfrak{A}', c', g', d', a', l', r', \mathfrak{L}', \mathfrak{R}', \hat{L}^{A'}, \hat{R}^{A'}$ in the Ann-category \mathcal{A}' .

Proof. The proof is completely similar to the one of Proposition 2.2. □

Following is the main result of this paper.

Theorem 3.3. Let Y_1, Y_2, \dots, Y_n be expressions of the limited family of objects $(X_i)_{i \in I}$ of an Ann-category \mathcal{A} . Let $\varphi_{i,i+1} : Y_i \rightarrow Y_{i+1}$ ($i=1, 2, \dots, n-1$), $\varphi_{n,1} : Y_n \rightarrow Y_1$ be isomorphisms built from the isomorphisms $\mathfrak{A}, c, g, d, a, l, r, \mathfrak{L}, \mathfrak{R}, \hat{L}^A, \hat{R}^A$, the identification and laws \otimes, \oplus . Then, the diagram (2.1) commutes.

Proof. From the theorem 2.4 [9], the Ann-category \mathcal{A} can be faithfully embedded into a quite strict Ann-category \mathcal{A}' by the faithful Ann-functor $(F, \check{F}, \tilde{F})$. Moreover, (F, \check{F}) is compatible with the unitivity constraints. In order to prove the diagram (2.1) commutative, we consider its image under F

$$\begin{array}{ccccccc}
 & & & & & & F(\varphi_{n,1}) \\
 & & & & & & \downarrow \\
 & & & & & & FY_1 \xrightarrow{F(\varphi_{1,2})} FY_2 \rightarrow \dots \rightarrow FY_{n-1} \xrightarrow{F(\varphi_{n-1,n})} FY_n \\
 & & & & & & \downarrow \\
 & & & & & & f_1 \quad f_2 \quad \quad \quad f_{n-1} \quad f_n \\
 & & & & & & \downarrow \\
 & & & & & & Y'_1 \xrightarrow{\varphi'_{1,2}} Y'_2 \rightarrow \dots \rightarrow Y'_{n-1} \xrightarrow{\varphi'_{n-1,n}} Y_n \\
 & & & & & & \downarrow \\
 & & & & & & \varphi'_{n,1} \\
 & & & & & & \uparrow
 \end{array}$$

where f_i are canonical isomorphisms, whereas $\varphi_{i,i+1}$ ($i = 1, 2, \dots, n-1$), $\varphi_{n,1}$ are morphisms making regions from (1) to (n) and the parameter commute according to the Prop.3.2. These morphisms are built from isomorphisms $c', \mathfrak{L}', \hat{L}^A, id$ and by laws \otimes, \oplus . Applying Theorem 2.3, the region (b) commutes. This implies that the region (a) commutes. This completes the proof. \square

Remark. The coherence theorem can be stated in another way as follows: Between two objects of the Ann-category \mathcal{A} , there exists no more than one morphism built from morphisms $\mathfrak{A}, c, g, d, a, l, r, \mathfrak{L}, \mathfrak{R}, \hat{L}^A, \hat{R}^A$ and laws \otimes, \oplus .

$$\begin{array}{ccc}
 (\sum_I A_i)(\sum_J B_j)(\sum_T C_t) & \xrightarrow{id \otimes F_{J,T}} & (\sum_I A_i)(\sum_{J \times T} B_j C_t) \\
 \downarrow id & & \downarrow id \\
 (\sum_I A_i)(\sum_J B_j)(\sum_T C_t) \oplus A_\alpha(\sum_J B_j)(\sum_T C_t) & \xrightarrow{(id \otimes F_{J,T}) \oplus (id \otimes F_{J,T})} & (\sum_{I'} A_i)(\sum_{J \times T} B_j C_t) \oplus A_\alpha(\sum_{J \times T} B_j C_t) \\
 \downarrow (F_{I',J} \otimes id) \oplus (F_{A_\alpha} \otimes id) & & \downarrow F_{I',J \times T} \oplus f_{A_\alpha} \\
 (\sum_{I' \times J} A_i B_j)(\sum_T C_t) \oplus (\sum_J A_\alpha B_j)(\sum_T C_t) & \xrightarrow{F_{I' \times J, T} \oplus F_{J, T}} & \sum_{I' \times J \times T} A_i B_j C_t \oplus \sum_{J \times T} A_\alpha B_j C_t \\
 \downarrow id & & \downarrow id \\
 (\sum_{I' \times J} A_i B_j \oplus \sum_J A_\alpha B_j)(\sum_T C_t) & \xrightarrow{F_{I \times J, T}} & \sum_{I \times J \times T} A_i B_j C_t
 \end{array}$$

$F_{I,J} \otimes id$ (left side), $F_{I,J \times T}$ (right side)

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